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HOMOCLINIC ORBITS FOR A CLASS OF HAMILTONIAN SYSTEMS

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ABSTRACT

We establish the existence of a homoclinic solution of the Hamiltonian system

$$(*) \qquad \ddot{q} + V_q(t, q) = 0$$

assuming that the potential V is T periodic in t, grows more rapidly than quadratically as $|q| \to \infty$ and satisfies some other technical conditions. The homoclinic solution is obtained as the limit of subharmonic solutions of $(*)$. The subharmonic solutions are found using a *minimax* argument.

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Homoclinic orbits for a class of Hamiltonian systems

Paul H. Rabinowitz

Introduction

There is a large literature on the use of variational methods to prove the existence of periodic solutions of Hamiltonian systems. However it is only relatively recently that these methods have been applied to the existence of homoclinic or heteroclinic orbits of Hamiltonian systems. See [1-5]. Such orbits have been studied since the time of Poincaré but mainly by perturbation methods.

Our goal in this paper is to prove the existence of homoclinic orbits for the second order Hamiltonian system:

$$\text{(HS)} \qquad \ddot{q} + V_q(t,q) = 0$$

where $q \in \mathbf{R}^n$ and V satisfies

(V_1) $V(t,q) = -1/2(L(t)q,q) + W(t,q)$ where L is a continuous T-periodic matrix valued function and $W \in C^1(\mathbf{R} \times \mathbf{R}^n, \mathbf{R})$ is T-periodic in t,

(V_2) $L(t)$ is positive definite symmetric for all $t \in [0,T]$,

(V_3) there is a constant $\mu > 2$ such that

$$0 < \mu W(t,q) \le (q, W_q(t,q))$$

for all $q \in \mathbf{R}^n \backslash \{0\}$, and

(V_4) $W(t,q) = o(|q|)$ as $q \to 0$ uniformly for $t \in [0,T]$.

Note that $(V_1) - (V_4)$ imply that $q(t) \equiv 0$ is a "trivial" homoclinic orbit of (HS). We will prove:

Theorem 1: If V satisfies $(V_1) - (V_4)$, (HS) possesses a nontrivial homoclinic solution, $q(t)$ emanating from 0 such that $q \in W^{1,2}(\mathbf{R}, \mathbf{R}^n)$.

Our study of (HS) was motivated by a recent paper of Coti-Zelati and Ekeland [1] which treated the first order Hamiltonian system.

$$(2) \qquad \dot{z} = JH_z(t,z) \qquad J = \begin{pmatrix} 0 & -id \\ id & 0 \end{pmatrix}$$

where $z \in \mathbf{R}^{2n}$. The function $H \in C^2(\mathbf{R} \times \mathbf{R}^{2n}, \mathbf{R})$ is T periodic in t with $H(t,z) = 1/2(Az,z) + R(t,z)$ where A is a constant symmetric matrix such that JA has no eigenvalues with 0 real part and R is strictly convex, satisfies V_3 (with $q \in \mathbf{R}^{2n}$), and $|R(t,z)| \le K|z|^\mu$ for all $z \in \mathbf{R}^{2n}$ for some $K > 0$. The convexity of R leads to a "dual" variational formulation of (2) which Coti-Zelati and Ekeland then study. Using a concentration compactness argument in the sense of P. L. Lions [6], they are able to apply a variant of the Mountain Pass Theorem to find a homoclinic orbit of (2).

Recently Hofer and Wysocki [5] have generalized the results of [1] by dropping the convexity conditions on H but also requiring that $|R_z(t,z)| \le K_1|z|^{\mu-1}$ for all $z \in \mathbf{R}^{2n}$. They use a rather different argument than [1] based on the study of certain first order elliptic systems that have also been useful in recent work on symplectic geometry. See e.g. [7].

Our approach to (HS) differs from both of the above. We will find the homoclinic solution q as the limit, as $k \to \infty$, of $2kT$ periodic solutions, q_k. The approximating solutions are obtained via the Mountain Pass Theorem. Then appropriate estimates for the critical values, c_k associated with q_k and on q_k allow us to pass to a limit to get q. The details will be carried out in §1.

§1. Proof of Theorem 2

For each $k \in \mathbf{N}$, let $E_k \equiv W^{1,2}_{2kT}(\mathbf{R},\mathbf{R}^n)$, the Hilbert space of $2kT$ periodic functions on $\mathbf{R}$ with values in $\mathbf{R}^n$ under the norm

$$\left(\int_{-kT}^{kT}(|\dot q(t)|^2 + |q(t)|^2)dt\right)^{1/2}$$

To exploit the form of (HS) and (V_2), it is more convenient to work with the equivalent norm

$$\|q\|_k^2 = \int_{-kT}^{kT}[|\dot q(t)|^2 + (L(t)q(t),q(t))]dt \tag{3}$$

Set

$$\begin{aligned} I_k(q) &= \int_{-kT}^{kT}[\frac{1}{2}|\dot q|^2 - V(t,q)]dt \\ &= \frac{1}{2}\|q\|_k^2 - \int_{-kT}^{kT} W(t,q)dt. \end{aligned} \tag{4}$$

Then $I_k \in C^1(E_k,\mathbf{R})$ and satisfies the Palais-Smale condition. See e.g. [8, Theorem 2.61]. Moreover critical points of I_k in E_k are classical $2kT$ periodic solutions of (HS). We will obtain a critical point of I_k by using a standard version of the Mountain Pass Theorem. Since the minimax characterization it provides for the critical value is important for what follows, we state the result precisely. Let $B_\rho(0)$ denote an open ball of radius ρ about 0.

Proposition 5 [9]: Let E be a real Banach space and $I \in C^1(E,\mathbf{R})$ satisfy the Palais-Smale condition. If further $I(0) = 0$,

(I_1) There exist constants $\rho, \alpha > 0$ such that

$$I\Big|_{\partial B_\rho(0)} \geq \alpha$$

and

(I_2) There exists $e \in E \backslash \overline{B_\rho}(0)$ such that $I(e) \leq 0$,
then I possesses a critical value $c \geq \alpha$ given by

$$c = \inf_{g \in \Gamma} \max_{s \in [0,1]} I_k(g(s)) \tag{6}$$

where

$$\Gamma = \{g \in C([0,1], E) | g(0) = 0 \quad \text{and} \quad g(1) = e\} \tag{7}$$

For our setting, clearly $I_k(0) = 0$. Moreover by (V_3),

$$W(t,\xi) \leq W(t, \frac{\xi}{|\xi|})|\xi|^\mu \quad \text{for} \quad 0 < |\xi| \leq 1 \tag{8}$$

$$\geq W(t, \frac{\xi}{|\xi|})|\xi|^\mu \quad |\xi| \geq 1$$

It is easy to see that there are constants $\beta_k, \gamma_k > 0$ such that

$$\beta_k \|q\|_{L^\mu[-kT,kT]} \leq \|q\|_{L^\infty[-kT,kT]} \leq \gamma_k \|q\|_k \tag{9}$$

for all $q \in E_k$. Therefore if $\|q\|_k \leq \gamma_k^{-1}$, $\|q\|_{L^\infty} \leq 1$ and

$$\int_{-kT}^{kT} W(t, q(t))dt \leq \int_{-kT}^{kT} W(t, \frac{q(t)}{|q(t)|})|q(t)|^\mu dt \leq \tag{10}$$

$$\leq \max_{t \in [0,T], \xi \leq 1} |W(t,\xi)| \|q\|^\mu_{L^\mu[-kT,kT]} \leq (\frac{\gamma_k}{\beta_k})^\mu \|q\|_k^\mu$$

Since $\mu > 2$, (10) shows

$$\int_{-kT}^{kT} W(t, q(t))dt = o(\|q\|_k^2) \quad \text{as} \quad q \to 0 \quad \text{in} \quad E_k \tag{11}$$

and

$$(12) \qquad I_k(q) = \frac{1}{2}\|q\|_k^2 + o(\|q\|_k^2)$$

as $q \to 0$ in E_k. Hence I_k satisfies (I_1) of Proposition 5. Moreover by (8) again, there are constants $a_1, a_2 > 0$ such that

$$(13) \qquad I_k(\beta\varphi) \leq \frac{1}{2}\beta^2\|\varphi\|_k^2 - a_1|\beta|^\mu \int_{-kT}^{kT} |\varphi|^\mu dt + a_2$$

for all $\beta \in \mathbf{R}$ and $\varphi \in E_k \backslash \{0\}$. Now (13) shows (I_2) holds with $e = e_k$, a sufficiently large multiple of any $\varphi \in E_k \backslash \{0\}$. Consequently by Proposition 5, I_k possesses a positive critical value c_k given by (6) and (7) with $E = E_k$ and $\Gamma = \Gamma_k$. Let q_k denote the corresponding critical point of I on E_k. Note that $q_k \not\equiv 0$ since $c_k > 0$.

The next step in the proof is to obtain k independent estimates for c_k and q_k. Let $\varphi \in E_1 \backslash \{0\}$ such that

$$(14) \qquad (i)\ \varphi(\pm T) = 0 \quad \text{and} \qquad (ii)\ I_1(\varphi) \leq 0$$

(By (13), if ψ satisfies (i), any sufficiently large multiple φ of ψ satisfies (i) and (ii)). Define

$$(15) \qquad \begin{aligned} e_k(t) &= \varphi(t), \ |t| \leq T \\ &= 0 \qquad T \leq |t| \leq kT. \end{aligned}$$

Then by (12), $e_k \in E_k \backslash \{0\}$ and $I_k(e_k) = I_1(e_1) \leq 0$. Note also that $g_k(s) = se_k \in \Gamma_k$ for all $k \in \mathbf{N}$ and $I_k(g_k(s)) = I_1(g_1(s))$. Therefore by (6),

$$(16) \qquad c_k \leq \max_{s \in [0,1]} I_1(g_1(s)) \equiv M$$

independently of k.

The estimate (16) leads to a priori bounds for q_k. Since $I'(q_k) = 0$, by (V_2),

$$(17) \qquad \begin{aligned} c_k &= I_k(q_k) - \frac{1}{2}I_k'(q_k)q_k \\ &= \int_{-kT}^{kT} [\frac{1}{2}(q_k, W_q(t, q_k)) - W(t, q_k)]dt \\ &\geq (\frac{\mu}{2} - 1) \int_{kT}^{kT} W(t, q_k)dt. \end{aligned}$$

Hence (4) and (17) yield a k-independent bound for $\|q_k\|_k$:

$$(18) \qquad \|q_k\|_k^2 = 2c_k + \int_{-kT}^{kT} W(t, q_k)dt \leq (2 + \frac{2}{\mu - 2})M \equiv M_1$$

A k-dependent bound on $\|q_k\|_{L^\infty[-kT,kT]}$ now follows from (9). Moreover a better estimate can be obtained as follows: For $q \in E_k$ and $t, \tau \in [-kT, kT]$,

$$|q(t)| \leq |q(\tau)| + |\int_\tau^t \dot{q}(s)ds|. \tag{19}$$

Integrating (19) over $[t - \frac{1}{2}, t + \frac{1}{2}]$ shows

$$\begin{aligned} |q(t)| &\leq \int_{t-\frac{1}{2}}^{t+\frac{1}{2}} |q(t)|d\tau + \int_{t-\frac{1}{2}}^{t+\frac{1}{2}} |\int_\tau^t \dot{q}(s)ds|d\tau \\ &\leq 2(\int_{t-\frac{1}{2}}^{t+\frac{1}{2}} (|\dot{q}(\tau)|^2 + |q(t)|^2)d\tau)^{1/2}. \end{aligned} \tag{20}$$

Hence

$$\|q\|_{L^\infty[-kT,kT]} \leq a_3\|q\|_k \tag{21}$$

where a_3 depends on L. Now (18) and (21) imply

$$\|q_k\|_{L^\infty[-kT,kT]} \leq a_3 M_1^{1/2} \equiv M_2 \tag{22}$$

with M_2 independent of k. Finally (HS) provides bounds for q_k in $C^2[-kT, kT]$ independent of k.

The C^2 bounds just obtained for q_k together with (HS) and (18) show a subsequence of q_k converges in $C^2_{loc}(\mathbf{R}, \mathbf{R}^n)$ to a solution q of (HS) satisfying

$$\int_{-\infty}^{\infty} [|\dot{q}|^2 + (Lq, q)]dt < \infty \tag{23}$$

It remains to show that $q \not\equiv 0$ and is a homoclinic solution of (HS). By (23),

$$\int_{|t|\geq m} [|\dot{q}|^2 + |q|^2]dt \to 0 \tag{24}$$

as $m \to \infty$. Hence by (20), $q(t) \to 0$ as $t \to \pm\infty$. If

$$\int_m^{m+1} |\ddot{q}|^2 dt \to 0 \tag{25}$$

as $m \to \pm\infty$, (20) with $\dot{q}$ replacing q and (24) imply $\dot{q}(t) \to 0$ as $t \to \pm\infty$. To verify (25), by (HS), (V_1), and (24), it suffices to show

$$\int_m^{m+1} |W_q(t, q)|dt \to 0 \tag{26}$$

as $m \to \pm\infty$. Since $W_q(t,0) = 0$ and it has already been shown that $q(t) \to 0$ as $t \to \pm\infty$, (26) follows.

Lastly we must show that $q \not\equiv 0$. Taking the inner product of (HS) with q_k and integrating by parts gives:

$$\|q_k\|_k^2 = \int_{-kT}^{kT} (q_k, W_q(t, q_k))dt. \tag{27}$$

Set $Y(0) = 0$ and for $s > 0$,

$$Y(s) = \max_{\substack{t\in[0,T] \\ |\xi|\leq s}} \frac{(\xi, W_q(t,\xi))}{|\xi|^2}. \tag{28}$$

Then by (V_3) and (8), it is easy to check that $Y \in C(\mathbf{R}^+, \mathbf{R}^+)$, $Y(s) > 0$ if $s > 0$, Y is monotone nondecreasing, and $Y(s) \to \infty$ as $s \to \infty$. The definition of Y shows

$$\frac{(q_k(t), W_q(t, q_k(t))}{|q_k(t)|^2} \leq Y(\|q_k\|_{L^\infty[-kT,kT]}) \tag{29}$$

for all $t \in [-kT, kT]$. Hence by (27) and (29),

$$\begin{aligned} \|q_k\|_k^2 &\leq Y(\|q_k\|_{L^\infty[-kT,kT]}) \int_{-kT}^{kT} |q_k|^2 dt \\ &\leq a_3 Y(\|q_k\|_{L^\infty[-kT,kT]}) \|q_k\|_k^2. \end{aligned} \tag{30}$$

Since $\|q_k\|_k > 0$,

$$Y(\|q_k\|_{L^\infty[-kT,kT]}) \geq \frac{1}{a_3}. \tag{31}$$

Consequently the properties of Y imply there is a $\beta > 0$ (and independent of k) such that

$$\|q_k\|_{L^\infty[-kT,kT]} \geq \beta \tag{32}$$

Now to complete the proof, observe that by the T-periodicity of L and W, whenever $p(t)$ is a $2kT$ periodic solution of (HS), so is $p(t + jT)$ for all $j \in \mathbf{Z}$. Hence by replacing $q_k(t)$ earlier if necessary by $q_k(t + jT)$ for some $j \in [-k, k] \cap \mathbf{Z}$, it can be assumed that the maximum of $|q_k(t)|$ occurs in $[0, T]$. Therefore if $q_k(t) \to 0$ in C^2_{loc} along our subsequence,

$$\|q_k\|_{L^\infty[-kT,kT]} = \max_{t\in[0,T]} |q_k(t)| \to 0 \tag{33}$$

contrary to (32). The proof is complete.

Remark 34: If V is independent of t, stronger results can be obtained by more direct arguments as will be shown in a joint paper with K. Tanaka.

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